Marginal Cost Pricing with Joint Cost: A Different Objective

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I. Introduction

Electrical energy in different periods of the day is a typical example of a joint product. An input of equipment provides capacity in technically fixed proportions during each period. However, it is not necessary that production in each period should be equal to capacity. Further, an item of durable capital equipment provides capacity over its lifetime in fixed proportions (perhaps unequal if there is physical deterioration) but it will not necessarily be optimal to maintain production at full capacity. This gives rise to the peak load pricing problem.

The solution to the problem involves the allocation of the joint costs to the separate products. To accomplish this an objective must be specified that is consistent with a set of operating criteria. The prime objective of electric utility operation is often stated as the maximization of social welfare. That this is a goal to be desired, few would question; that it cannot be translated directly into operational criterion for system operation, few would deny. Translation requires agreement on a definition for the deceptively simple phrase "social welfare" but also some assurance that the concept as defined is measureable. For these reasons the common expression chosen for social welfare is gross benefit less total cost which is equivalent to consumer's surplus plus total revenue less total cost. Consumer's surplus is simply the differences between what a consumer would be willing to pay for a good or service and what he actually pays.

A convenient formulation for the multi-dimensional consumer's surplus with dependent demand functions is given by Hotelling [1]. If one considers a set of unrelated commodities whose demand functions are

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(1) $p_i = p_i\ (q_1, q_2, \cdots, q_n),\ (i=1,2,\cdots,n)$

then the formulation is of the form

(2) $\int_c (p_1\, dq_1 + p_2\, dq_2 + \cdots + p_n\, dq_n) - \sum_{i=1}^{n} \overline{p}_i\ \overline{q}_i$

with the integration being performed along some curve c and P_i and q_i being the equilibrium price and quantity.

There are two difficulties with this formulation. First, the necessary conditions for a local maximum or minimum of a real function require that the first derivatives of that function vanish at all critical points. Since the consumer's surplus is a part of the objective function which is to be optimized the differentiability of consumer's surplus is required if classical optimization techniques are to be used. Second, consumer's surplus, as it is defined, depends on the particular path c chooses and is thus not unique.

Both problems can be solved by introducing a theorem on line integrals that states that if a set of integrability conditions are assumed, the coefficients of substitution or complementarity of any pair of goods (i and k) must equal the corresponding pair (k and i) of demand functions, i.e.,

(3) $\frac{\partial p_i}{\partial q_k} = \frac{\partial p_k}{\partial q_i}$

then a line integral of the form of relation (2) is differentiable and the value of the consumer's surplus is independent of the path c.[1] Thus, in order to accept the results of the peak load pricing solution based on the Hotelling formulation of consumer's surplus, the integrability conditions, (3), need to be assumed.

The viability of this assumption is empirical not theoretical. If one derives the system of demand functions from a utility function, direct or indirect, there is no problem of integrability—although there may still be restrictions on the set of demand functions (Lau [2]). The integrability problem arises if the point of departure of the analysis is the system of demand functions rather than the direct or the indirect utility functions.[2] There is no reason why integrability should be expected to hold with aggregate data (aggregated across individuals). With aggregate data, the conditions for integrability are extremely restrictive: either the preferences of all individuals

1 See Pressman [4, pp. 324-325].

2 This is the point of departure in peak load pricing formulations.

are all identical and homothetic; or the preferences and incomes of all individuals are identical. In practice, neither of these two conditions are likely to hold. In fact recent work has resulted in the rejection of the hypothesis of integrability (Lau [2]). Some reasons include: the error in aggregation across individuals (failure to take into account the changing distribution of income); the possibility of taste differences across individuals at any one time; the error in ignoring dynamic considerations (habit-formation, stock adjustment, etc.).

With a rejection of the hypothesis of integrability, where does this leave us? The common expression for social welfare must be rejected and an alternative inserted in its place. Whether the operating criteria will change as a result remains to be seem. This is the objective of this paper.

II. A Welfare Maximization Approach to Marginal Cost Pricing with Joint Costs

As a point of departure, the various operating rules resulting from the traditional approach to the problem will be illustrated. It will be assumed that we are not concerned with terminal conditions. This assumption is made solely for ease of explication and will not alter the development of the argument and the consequent results. For each time peiod t up to a fixed horizon T, let

y_t denote the quantity of electrical energy demanded,

$f_t(y_t)$ denote the demand function,

x_t^θ denote the production on equipment of vintage θ,

x_t denote total supply,

c_t^θ denote the marginal operation and maintenance cost (hereafter, operating cost) on equipment of vintage θ,

W_t denote the purchase of new equipment,

ξ_t denote the marginal purchase price of equipment,

r denote the rate of interest.

Present value of consumer's surplus plus total revenue less total cost, which is to be maximized, is given by

$$(4) \quad \sum_{t=1}^{T} \sigma^{t-1} \left(\int f_t(y_t)\, dy_t - \sum_{\theta=1}^{t} c_t^\theta x_t^\theta - \xi_t W_t \right)$$

where

$$\sigma^{t-1}=1/(1+r)^{t-1}.$$

The system operation is subject to several constraints:

(a) Total quantity demanded must be less than or equal to the total supply

(5) $y_t \leq x^t, \quad t=1,2,\cdots,T.$

(b) Total supply is less than or equal to the sum of production on equipment of all vintages

(6) $x_t \leq \sum_{\theta=1}^{t} x_t^{\theta}, \quad t=1,2,\cdots, T;$

(c) For each vintage of equipment, production cannot exceed capacity

(7) $x_t^{\theta} \leq W_{\theta}, \quad t=\theta, \theta+1,\cdots, T \text{ and } \theta=1, 2,\cdots, T.$

Now associate the dual variables $\sigma^{t-1}\rho_{1t}$, $\sigma^{t-1}\rho_{2t}$, and $\sigma^{t-1}\rho_t^{\theta}$

with constraints (5), (6) and (7), respectively.

Providing the demand curves are downward sloping, the objective function, (4), is concave and the Kuhn-Tucker conditions will give the necessary and sufficient conditions for a competitive solution. Multiplying through by σ^{t-1} where appropriate, the Kuhn-Tucker conlitions are

(8) $y_t \leq x_t \qquad \rho_{1t} \geq 0$

(9) $x_t \leq \sum_{\theta=1}^{t} x_t^{\theta} \qquad \rho_{2t} \geq 0$

(10) $x_t^{\theta} \leq W_{\theta} \qquad \rho_t^{\theta} \geq 0$

(11) $\rho_{1t} \leq \rho_{2t} \qquad x_t \geq 0$

(12) $f_t(y_t) \leq \rho_{1t} \quad y_t \geq 0$

(13) $\rho_{2t} \leq c_t^{\theta} + \rho_t^{\theta}$ $\quad x_t^{\theta} \geq 0$

(14) $\sum_{t' \geq t} \sigma^{t'-t} \rho_{t'}^{\theta} \leq \xi_t \quad W_t \geq 0$

ρ_t^{θ} can be interpreted as the quasi-rent of equipment of vintage θ in period t and, if that equipment is utilized, is equal to supply price minus marginal operating cost. Thus,

(13′) if $x_t^{\theta} > 0$, then $\rho_t^{\theta} = \rho_{2t} - c_t^{\theta}$

Supposing in addition $y_t > 0$ and $x_t > 0$, the quasi-rent will be exactly equal to the market demand price ρ_{1t}, less the operating cost. If the equipment is not fully utilized, $x_t^{\theta} < W_{\theta}$, then its quasi-rent is zero. Thus,

(10′) if $x_t^{\theta} < W_{\theta}$, then $\rho_t^{\theta} = 0$

To minimize total cost, production will be scheduled on equipment in increasing order of marginal operating cost. Thus, once again assuming $y_t > 0$ and $x_t > 0$, the demand price in any period t equals marginal operating cost on the least efficient equipment then in use whose vintage is denoted by θ^{*}.

Thus,

(15) if $0 < x_t^{\theta^{*}} < W_{\theta^{*}}$, then $f_t(y_t) = c_t^{\theta}$

The quasi-rent of any equipment in use is in fact the difference between its marginal operating cost and that of the least efficient equipment then in operation. Thus,

(16) if $x_t^{\theta} > 0$, then $\rho_t^{\theta} = c_t^{\theta^{*}} - c_t^{\theta}$

conditional upon y_t and x_t being positive.

Equipment should be purchased in any period up to the point where the discounted sum of its quasi-rents equals marginal purchase price. Thus,

$$(14') \quad \text{if } W_\theta > 0, \text{ then } \sum_{t' \geq t} \sigma^{t'-t} \rho^\theta_{t'} = \xi_t$$

The quasi-rent of equipment in any period t can be regarded as that part of the output price set aside to cover depreciation and interest on capital. This is what Turvey [5, p. 291] called amortization. If new equipment is purchased in any period, demand price in that period is equal to marginal operating cost plus amortization on the new equipment. Thus,

$$(17) \quad \text{if } x^\theta_t > 0 \text{ and } y_t > 0, \; f_t(y_t) = c^t_t + \rho^t_t$$

Equipment of vintage θ should begin to be retired when its marginal operating cost rises above this level. Thus,

$$(18) \quad \text{if } c^\theta_t > c^t_t + \rho^t_t, \text{ then } x^\theta_t = 0.$$

Equation (14') can be interpreted to mean that present value of amortization equals the purchase price of equipment. Consequently, the present value of total profit is zero. The issue to be emphasized is that the amount set aside for amortization varies from period to period depending on demand and perhaps is equal to zero in some periods. With demand varying, a constant poportion of capacity cost each period should not necessarily be recouped. Rather, price should be set to just fully utilize capacity (conditional upon price not being below marginal production cost), and capacity should be selected so that over the life of the equipment, capacity cost is just recouped.

The foregoing then are the operating criteria for a public utility maximizing social welfare defined in expression (4). All of the results are conditional upon the integrability conditions obtaining. With the dubious validity of these conditions, it is necessary to posit an alternative objective while at the same time insuring a competitive equilibrium solution will result.

Before looking at this alternative let us define precisely what is meant by a competitive equilibrium solution in the current context.

III. Competitive Equilibrium Defined

An economic state is said to be in a competitive equilibrium if the following conlitions are met:

(i) market equilibrium condition where there is homogeneity and uniqueness of the market demand price, ρ^{1t} , and the market supply price, ρ_{2t} , and the quasi-rent, ρ_t^θ for each t and θ and no excess demand or excess supply poossibility

$$y_t - x_t \leq 0 \text{ and } \rho_{1t}(y_t - x_t) = 0$$

$$x_t - \sum_{\theta=1}^{T} x_t^\theta \leq 0 \text{ and } \rho_{2t}\ (x_t - \sum_{\theta=1}^{T} x_t^\theta) = 0$$

$$x_t^\theta \leq W_\theta \text{ and } \rho_t^\theta\ (x_t^\theta - W_\theta) = 0 \text{ for all } t \text{ and } \theta;$$

(ii) Optimum consumption equilibrilm condition

$$\rho_{1t} - f_t(y_t) \geq 0 \text{ and } y_t(\rho_{1t} - f_t(y_t)) = 0,\ y_t \geq 0\ ;$$

(iii) Optimum production condition

$$\rho_{2t} - c_t^\theta - \rho_t^\theta \leq 0 \text{ and } x_t^\theta(\rho_{2t} - c_t^\theta - \rho_t^\theta) = 0,\ x_t^\theta \geq 0;$$

(iv) Optimum investment condition

$$\sum_{t' \geq t} \sigma^{t'-t}\ \rho_{t'}^\theta - \xi_t \leq 0 \text{ and } W_\theta(\sum_{t' \geq t} \sigma^{t'-t}\ \rho_{t'}^\theta - \xi_t) = 0,\ W_\theta \geq 0\ ; \text{ and}$$

(v) Price equilibrium

$$\rho_{1t} - \rho_{2t} \leq 0 \text{ and } x_t(\rho_{1t} - \rho_{2t}) = 0,\quad x_t > 0.$$

At issue is the formulation of the problem so that these conditions are met. The results will yield the sought after competitive equilibrium conditions for peak load pricing.

IV. An Alternative Approach to Marginal Cost Pricing with Joint Costs

An alternative objective function with considerable intuitive appeal that does not possess the limitations inherent in the consumer's surplus expression (2) is net revenue. Net revenue is the present value of total revenue less total cost

$$(19) \quad NR = \sum_{t=1}^{T} \sigma^{t-1} (y_t \; f_t \; (y_t) - \sum_{\theta=1}^{t} c_t^{\theta} \; x_t^{\theta} - \xi_t \; W_t)$$

The net revenue maximum problem is formulated as Problem 1: find Ψ (y, x, X, W, ρ) that maximizes the net revenue function (19) subject to

(a) $\quad f_t(y_t) - \rho_{1t} \leq 0$

(b) $\quad \rho_{2t} - c_{\theta t} - \rho_t^{\theta} \leq 0$

(c) $\quad \rho_{1t} - \rho_{2t} \leq 0$

(d) $\quad \sum_{t' \geq t} \sigma^{t'-t} \; \rho_{t'}^{\theta} - \xi_t \leq 0$

(e) $\quad y_t - x_t \leq 0$

(f) $\quad x_t - \sum_{\theta=1}^{T} x_t^{\theta} \leq 0$

(g) $\quad x_t^{\theta} \leq W_{\theta}$

(h) $\quad y_t \geq 0, \; x_t^{\theta} \geq 0, \; x_t \geq 0, \; W_{\theta} \geq 0, \; \rho_{1t} \geq 0, \; \rho_{2t} \geq 0,$ and $\rho_t^{\theta} \geq 0$

for all θ and t, and

where

y is the set of all y_t

x is the set of all x_t^θ

X is the set of all x_t

W is the set of all W_θ , and

ρ is the set of all ρ_{1t}, ρ_{2t}, and ρ_t^θ

This problem does not possess the nice property that the feasibility set is necessarily non-empty due to the first two constraint sets, (a) and (b). Thus, the only proof given here is that if a solution exists for this problem, the solution satisfies competitive equilibrium stated above. For the following Lagrangean:

(20)

$$\phi\ (y, x, X, W, \rho, \eta, P) = \sum_{t=1}^{T} \sigma^{t-1} [y_t\ f_t(y_t) - \sum_{\theta=1}^{T} c_t^\theta\ x_t^\theta - \xi_t\ W_t]$$

$$+ \sum_{t=1}^{T} \sigma^{t-1} [\eta_{1t}\ (\rho_{1t} - f_t(y_t)) + \eta_{2t}(\rho_{2t} - \rho_{1t})$$

$$+ \sum_{\theta=1}^{T} \eta_{\theta t}(c_t^\theta + \rho_{t'}^\theta - \rho_{2t})$$

$$+ \sum_{\theta=1}^{T} \eta_t^\theta\ (\xi_t - \sum_{t' \geq t} \sigma^{t'-t}\ \rho_{t'}^\theta)]$$

$$+ \sum_{t=1}^{T} \sigma^{t-1} [P_{1t}(x_t - y_t) + P_{2t}(\sum_{\theta=1}^{T} x_t^\theta - x_t)$$

$$+ \sum_{\theta=1}^{T} P_t^\theta (W_\theta - x_t^\theta)]$$

where η is the set of all $\eta_{1t}, \eta_{2t}, \eta_{\theta t}, \eta_t^{\theta}$ and P is the set of all P_{1t}, P_{2t} and P_t

Providing the demand curves are downward sloping, the revenue function, (19), is concave and the Kuhn-Tucker conditions will give the necessary and sufficient conditions for an optimum. After elimination of the discount factor, they are

(21)	$y_t \leq x_t$	$P_{1t} \geq 0$
(22)	$x_t \leq \sum_{\theta=1}^{T} x_t^{\theta}$	$P_{2t} \geq 0$
(23)	$x_t^{\theta} \leq W_{\theta}$	$P_t^{\theta} \geq 0$
(24)	$P_{1t} \leq P_{2t}$	$x_t > 0$
(25)	$f_t(y_t) + y_t f_t'(y_t) - \eta_{1t} f_t'(y_t) \leq P_{1t}$	$y_t \geq 0$
(26)	$P_{2t} \leq c_t^{\theta} + P_t^{\theta}$	$x_t^{\theta} \geq 0$
(27)	$\sum_{t' \geq t} \sigma^{t'-t} P_{t'}^{\theta} \leq \xi_t$	$W_t \geq 0$
(28)	$f_t(y_t) \leq \rho_{1t}$	$\eta_{1t} \geq 0$
(29)	$\rho_{1t} \leq \rho_{2t}$	$\eta_{2t} \geq 0$
(30)	$\rho_{2t} \leq c_t^{\theta} + \rho_t^{\theta}$	$\eta_{\theta t} \geq 0$
(31)	$\sum_{t' \geq t} \sigma^{t'-t} \rho_{t'}^{\theta} \leq \xi_t$	$\eta_t^{\theta} \geq 0$

Conditions (21) through (27) are exactly the same as conditions (8) through (14) of the preceeding section with the exception of condition (25). The interpretation of the conditions is identical. Conditions (28) through (31) are just those necessary to provide for optimum consumption, price equilibrium, optimum production, and optimum investment.

It is interesting to note that the objective function. (19), is non-positive as long as the variables are subject to the constraint set (a) through (h) found following the statement of the problem. That is,

(32)

$$NR = \sum_{t=1}^{T} \sigma^{t-1} \left(y_t\ f_t(y_t) - \sum_{\theta=1}^{T} c_t^\theta\ x_t^\theta - \xi_t\ W_t\right)$$

$$\leq \sum_{t=1}^{T} \sigma^{t-1} \left(\sum_{\theta=1}^{T} x_t^\theta\ \rho_{1t} - \sum_{\theta=1}^{T} c_t^\theta\ x_t^\theta - \sum_{\theta=1}^{T} x_t^\theta\ \rho_t^\theta\right)$$

$$= \sum_{t=1}^{T} \sigma^{t-1} \left(\sum_{\theta=1}^{T} x_t^\theta\ \rho 1t - \sum_{\theta=1}^{T} (c_t^\theta + \rho_t^\theta) x_t^\theta\right)$$

$$\leq \sum_{t=1}^{T} \sigma^{t-1} \left(\sum_{\theta=1}^{T} (\rho_{1t} - \rho_{2t})\ x_t^\theta\right)$$

$$= \sum_{t=1}^{T} \sigma^{t-1} (\rho_{1t} - \rho_{2t})\, x_t \leq 0$$

The first inequality is obtained from (a), (d), (e), (f), and (g). The second inequality is obtained from (b). The last inequality is obtained from (c). The implications of this non-positiveness of the objective function is that if a solution exists for the problem, then the maximum value of the objective function must be zero.

The dual of Problem 1 can be specified as Problem 2: Find (η, P) that minimizes

$\phi(y, x, X, W, \rho, \eta, P)$ as given in equation (20).

Write the feasibility sets of Problem 1 and Problem 2 as R and S respectively. Then we have the result

(33) $\max_{R} \Psi(y, x, X, W, \rho) = \min_{S} \phi(y, x, X, W, \rho, \eta, P)$,

If a solution exists for either of the two problems (Luenberger [3], pp. 223-225.

However, since

(34) $\Psi_{\text{on } R}\ (y, x, X, W, \rho) \leq 0$

and relation (33) holds, we get

$$(35)\quad \underset{\text{on R}}{\Psi}(y, x, X, W, \rho) \leq 0 \leq \underset{\text{on S}}{\phi}(y, x, X, W, \rho, \eta, P)$$

Further, if $(\bar{y}, \bar{x}, \overline{X}, \overline{W}, \bar{\rho})$ solves Problem 1 then there exists a $(\bar{\eta}, \overline{P})$ such that

$$(36)\quad \Psi(\bar{y}, \bar{x}, \overline{X}, \overline{W}, \bar{\rho}) = \phi(\bar{y}, \bar{x}, \overline{X}, \overline{W}, \bar{\rho}, \bar{\eta}, \bar{P})$$

which, due to (35), reduces to

$$(37)\quad \Psi(\bar{y}, \bar{x}, \overline{X}, \bar{W}, \bar{\rho}) = \phi(\bar{y}, \bar{x}, \overline{X}, \bar{W}, \bar{\rho}, \bar{\eta}, \bar{P}) = 0$$

Therefore, if we assume that $(\bar{y}, \bar{x}, \overline{X}, \bar{W}, \bar{\rho})$ is a solution for Problem 1, then

$$(38)\quad \Psi(\bar{y}, \bar{x}, \overline{X}, \bar{W}, \bar{\rho}) = 0$$

follows immediately.

By making use of the primal-dual method (Luenberger [3, p. 299]), we can conclude that the dual solution set for Problem 2, consisting of $(\bar{\eta}, \bar{\rho})$, is exactly equal to the primal solution set consisting of $(\bar{y}, \bar{x}, \overline{X}, \bar{W}, \bar{\rho})$. That is, the solution set $(\bar{\eta}, \bar{\rho})$ is an exact replica of $(\bar{y}, \bar{x}, \overline{X}, \bar{W}, \bar{\rho})$.

Given this extremeiy interesting and simplifying property of the solution set, it is easy to justify the statement that the solution $(\bar{y}, \bar{x}, \overline{X}, \bar{W}, \bar{\rho})$ satisfies the conditions stipulated for a competitive equilibrium as difined above.

Thus, it has been proved that if the net revenue maximum problem yields a solution, the solution is a competitive price equilibrium solution and the operating criteria are identical to those under the welfare maximization approach involving consumer's surplus in the objective function.

V. Conclusion

If we adopt net revenue as the appropriate definition of social welfare to be maximized, the solution, if it exists, will yield a competitive equilibrium. This approach, though more demanding computationally, allows us to circumvent the problems associated with the maximization of an expression involving consumer's surplus.

Further, the net revenue maximization problem is a more general and pragmatic framework that when solved will attain an equilibrium for not only the non-integrable case but the case where the integrability conditions are assumed to hold as well. It should be emphasized that in this formulation, a tangible objective such as profit is used rather than an intangible objective such as consumer's surplus.

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