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ENDOGENOUS EQUILIBRIUM GROWTH WITH RECURSIVE PREFERENCES AND INCREASING RETURNS

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This paper studies the short-run and long-run behavior of a competitive economy in which both the discount factor and technological change are endogenously determined. In particular, the effect of saving behavior on persistent economic growth is considered. The paper provides the sufficient condition on the rate of impatience and the concavity of private technology for uniqueness and local stability of the steady state in the competitive economy. The condition implies determinacy of transitional equilibrium paths. However, the paper also shows that the presence of externalities and adjustment cost of investment can lead to indeterminacy of steady state equilibra.

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1. INTRODUCTION

This paper describes the dynamics of capital accumulation in a competitive economy in which consumer preferences are intertemporally dependent and technological changes are endogenous. The two main appealing features of the economy are a non-decreasing returns technology and recursive preference. The technology follows from the recent trend of the endogenous growth theory in which the economy can potentially grow in the long run. Recursive preferences allow for interdependence of utility over time and thus relax the common, but restrictive time-additivity of preferences. Based on a closed variant of Romer's (1986) technology with Uzawa's (1968) preferences, the paper characterizes an equilibrium in the competitive economy with flexible time discounting with non-decreasing returns technology. Of particular interest is to establish the uniqueness or determinacy of a competitive equilibrium and to study its local stability

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property in a growing economy.

While endogenous growth theory provides the conditions for a persistently growing economy without requiring exogenous technological change, most analysis has proceeded with the assumption of the additively time-separable utility, which implies rigid time preferences (Romer (1986), Lucas (1988)). This additive utility specification may yield results that seem strange in ordinary circumstances. Notably, Becker (1980) suggests that, in economies having a (asymptotically) linear technology (Rebelo (1990), Jones and Menualli (1990)) and thereby exhibiting a constant interest rate, a representative consumer will try either to save without limit or to borrow without limit, except in the knife-edge case that the rate of impatience equals the rate of interest. These economies then imply paradoxically that saving rates and rates of economic growth are inversely related. The constant discount rate hypothesis in the standard neo-classical technology also suggests unappealing implications for the trade in the world market, where all capital stocks eventually flow to the country whose rate of time preference is the lowest and thus other countries can no longer grow in the long run.

Escaping these dilemmas, this paper explicitly allows intertemporal dependence of consumption-saving choices by introducing recursive preference.[1] The class of recursive preferences shares many of the important properties of the additive preference family (e.g., Koopmans (1965), Uzawa (1968)), but incorporates certain richer characteristics: in particular, a set of possibilities for intertemporal substitution due to consumption-income changes over time. With time-separable preferences, a consumption-saving decision is based on usual income and substitution effects: The substitution effect depends on the productivity of investment. Decreasing returns as in a neo-classical growth model discourages saving, so this effect is negative. In the new growth theory,[2] this substitution effect will be non-negative due to non-decreasing returns technology. The income effect comes from the concavity of preference in which a consumer wants to smooth out consumption over the infinite horizon. When the future income is expected to be higher than the present income, then the income effect is positive. The balance of these effects determines consumption and saving in the time-additive case. In the case of recursive preferences, these two effects are supplemented by a third effect: The time path of consumption-income determines the marginal rate of impatience and this creates an additional intertemporal substitution effect in the saving decision.

The objectives of the paper are to characterize the dynamics and to establish the

[1] For example, recursive utility has been applies to explain questions on the distribution of income among heterogeneous households (Lucas and Stokey (1984), Epstein (1987)), on international trade (Obstfeld (1982)), on capital taxation (Chamley (1986)), on finance (Duffie and Epstein (1992)) and many others.

[2] In endogenous growth theory, the production function exhibits the (asymptotic) linear technology (e.g., Rebelo (1990), Jones and Menualli (1990)) and increasing returns technology (e.g., Romer (1986), Lucas (1988)).

stability condition for a competitive economy with endogenous time preference and technological changes. Prior to reaching these objectives, the paper explores the competitive existence condition on recursive preference and increasing returns technology.[3] In the presence of externalities in the competitive economy, the concavity conditions on preferences (a felicity and discounting function), and technology (a private production and investment function) ensure existence of a competitive equilibrium. This generalizes the Brock-Gale's joint condition on preference and technology in a neo-classical competitive economy to a growing competitive economy with flexible time preferences.

In addition, the paper also establishes the sufficient conditions for a unique steady state. The uniqueness property comes essentially from the flexibility of time preference, along with the concavity condition for the existence of the competitive equilibrium in relation with the effects of externalities. Furthermore, we also show that this condition is sufficient for saddle stability and thereby uniqueness of an equilibrium trajectory. Hence, this extends the result of Benhabib and Gali (1995) that a unique steady state induces the uniqueness of a transitional equilibrium path.

The stability condition in this paper clarifies how the artifice of a fixed rate of impatience affects the stability of the competitive equilibrium. In time-separable models with either linear or increasing returns technology, a steady state, if one exists, is always locally unstable.[4] This is because both the income and the substitution effect are positive in those models. But, together with the concavity condition of the production function in terms of private capital stocks, an increase in a rate of time preferences plays an important role as a stabilizer.[5] More precisely, when a consumer becomes impatient, the intertemporal substitution effect from consumption-saving choices becomes negative. Therefore if this negative income effect dominates the net of the usual income and substitution effects, then capital accumulation is stabilized over time.

The present paper also provides a source of multiplicity of steady states and indeterminacy of equilibria. Even though externality is a main source of both multiplicity and indeterminacy as in the literature,[6] adjustment cost of investment also plays a critical role for indeterminacy of equilibria (see Benhabib and Peril (1994)). I

[3] In the usual case of non-convexity of a resource feasible set, a social optimum and a competitive equilibrium fails to exist when technology exhibits increasing returns to scale.

[4] In addition, sunspots equilibrium and business cycles can be constructed in increasing returns models. For related topics, refer to the symposium issue of *Journal of Economic Theory* (1994).

[5] The impatience assumption on recursive preferences has been wildly used and its economic meaning and implications have been discussed in the literature (for example, refer to Koopmans (1960), Epstein (1987), and Lucas and Stokey (1984)).

[6] In models of increasing returns technology with constant time preference, recent research has reported multiple steady states and indeterminacy of equilibrium paths (e.g., Howitt and McAfee (1988), Boldrin and Rustichini (1994), Benhabib and Perli (1994)).

also show that the indeterminacy does not require a sufficiently large the effect of externalities as long as the adjustment cost is large enough.

There are important contributions on long-run growth with endogenous rate of time preference. Notably, Farmer and Lahiri (2005) and Palivos *et al.* (1997) study a model of homogeneity of recursive preferences for a balanced growth with convex technology as in Jones and Manuelli (1990). Drugeon (1998) concerns a model with Romer's (1996) technology and Uzawa's (1968) preference. However, their contributions differ from those in this paper at least in the three aspects: First, this paper relates uniqueness of a steady state with local saddle stability of an equilibrium path. The paper then explains these properties in terms of the income and substitution effects. Second, the paper provides the sufficient condition for indeterminacy of steady state equilibrium paths. That is, there is a continuum of competitive equilibria converging to one of steady states provided that many steady states exist. Third, the results in this paper are in contrast with indeterminacy results in the literature including Drugeon (1998). In particular, the adjustment cost of investment together with (not necessarily large) externality is recognized as a source of the indeterminacy.

This paper is organized as follows. The model is described in Section 2. In Section 3, the Hamiltonian equations for the competitive economy are derived and conditions for a positive price and the capital and consumption policy function are examined. Section 4 examines uniqueness and local stability in the competitive economy. Also, the possibility of indeterminacy and instability is discussed. The final section contains concluding remarks.

2. THE MODEL

The economy is perfectly competitive and the setup is a dynamic general equilibrium model. There is no uncertainty in the economy. Time is continuous on an infinite horizon. There is a single consumption good available at each instant up to an infinite horizon. A consumption path is represented by C. The t-th period consumption level corresponding to C is denoted by $c(t)$. Suppose that $c(t)$ is a continuously differentiable map from $\Re_+$ to $\Re_+$ over t. The programming problem is defined for a given felicity function $u(c)$ and discounting function $v(c)$. The recursive objective functional $U(C)$ after Uzawa (1968) is given by

$$U(C) = \int_0^\infty u(c(t)) \exp[-\int_0^t v(c(s))ds]dt \,.$$

In the interest of tractability and simplicity, the following restrictions on the felicity and discounting functions are imposed to capture the main features of an endogenous saving rate in a competitive economy.

Assumptions on the Felicity and the Discounting Functions:

(F1) $u:\Re_+ \to \Re_+$ and $v:\Re_+ \to \Re_+$ are twice continuously differentiable on $\Re_+$,
(F2) u and v are increasing and concave in c,
(F3) there exists $\zeta \geq 0$ such that $u''/u' - \zeta > v''/v'$,[7]
(F4) there exists d such that $v \geq d > 0$.

The first assumption (F1) is only for convenience. Assumption (F1), together with Assumption (F2), allows the duality argument to be applied using a conjugate function (e.g., Araujo and Scheinkman (1977)). The last part of (F2) is imposed to ensure the existence of a solution to the consumer's optimization problem. The sole purpose of (F3) is to guarantee a positive equilibrium price of capital. (The following section discusses this issue in detail.) Obviously, Assumption (F4) excludes the possibility of undiscounting the future. For example, when $\eta, \chi \in \Re_+$, $A_1, A_2 \in \Re_-$ and $-\eta - \zeta > -\chi$, $u(c) = A_1 e^{-\eta c}$ and $v(c) = d + A_2 e^{-\chi c}$ satisfy the above assumptions for the facility and discounting function, respectively.

Technology is a close variant of that in Romer (1986) and Lucas (1988). Let $\hat{f}_j(k_j, z, x_j)$ denote the instantaneous rate of output for firm j as a function of its capital stock k_j, economy-wide capital stock z, and the level of all other inputs x_j. The capital stock z is formed by learning-by-doing as a side product of individual investment. Then, given $z(t)$ for each t, assume that $\hat{f}_j(k_j, z, x_j)$ exhibits constant returns to scale in (k_j, x_j). It is also assumed that z is a nonrivalry good that each firm can access at zero cost. Let z be contributed by each individual firm's own investment and measured in terms of the aggregate capital stock in the economy, i.e., $z(t) = \sum_{j=1}^{N} k_j(t)$ where N is the number of firms in this economy. For the sake of simplicity I assume that all firms are identical. Then $z(t) = Nk(t)$. I can further assume $N = 1$.[8] This does not affect any results in this paper because every firm in the economy

[7] The first-order and second-order derivative of a function $h(x):\Re \to \Re$ is denoted by $h'(x)$ and $h''(x)$, respectively.

[8] Hence, $z(t) = k(t)$ in the equilibrium path, but not in the firm's problem, because $z(t)$ is externalities to the firm in the competitive economy. This setup is identical to Romer (1986). However, we may point out that a Nash-type mechanism of externalities is not so appealing when the firm knows that there exists no other firm in the market. (I thank the anonymous referee for this point.) For clarity, we may take $N > 1$ throughout the paper. But, it is also clear that keeping that $z(t) = Nk(t)$ will not change any of arguments in this paper.

exhibits constant returns to (k_j, x_j). In addition, factors other than capital are assumed to be supplied in fixed quantities, so that $x(t)$ is constant. We can normalize it to 1. It can be thought of as a constant supply of labor. Therefore, a representative firm's production function $\hat{f}_j(k_j, z, x_j)$ can be simplified: formally $f(k,z) \equiv \hat{f}_j(k_j, z, 1)$.

Additional capital stocks can be produced by forgoing consumption. Let $\dot{k} = G(I,k)$ where $I \equiv f(k,z) - c$ is the investment function for an individual firm. Under Assumption (T4) below, $G(I,k)$ can be rewritten in terms of the proportional rate of growth such that $g(I/k) \equiv G(I/k, 1)$. Therefore, changes in the capital stock can be expressed as $\dot{k} = kg(I/k)$. It is worthwhile noting that the trade-off between current consumption and capital is no longer one-for-one.

As in Romer (1986), the following assumptions on technology are imposed to capture the main features of a competitive economy with externalities.

Assumptions on Technology:

(T1) $f: \Re_+^2 \to \Re_+$ and $G: \Re_+^2 \to \Re_+$ are twice continuously differentiable,
(T2) for any z, $f(k,z)$ is concave in the first argument k,
(T3) $f(k,k) \le \mu + k^{\rho}$ where $\mu, \rho \in \Re_+$ and $0 \le \rho$,
(T4) $G(I,k)$ is concave and homogeneous of degree one,
(T5) $g(I/k) \in [0, \alpha]$ with $g(0) = 0$ and $g'(0) = 1$ where $\alpha \in \Re_+$.

The first assumption (T1) is made for the sake of convenience. The second assumption (T2) is required, along with concavity of $G(I,k)$ in (T4), for the existence of a solution to the firm's profit maximization problem within an endogenous growth model.[9] The third assumption (T3) allows the economy to face an increasing (as well as decreasing) returns production function. The last part of (T4) is for convenience. The last assumption (T5) depicts the nature of an increase in the cost of investment as physical (or human) capital increases. This cost structure has been recognized in, for example, the cost of higher education or research and development (Romer (1986)); of 'time to build' (Kydland and Prescott (1982)); and of technology transfer among countries, etc. The following functions demonstrate the above technology: Under $0 < \omega \le 1$, $0 < \vartheta$, $\omega + \vartheta = \rho$, $\omega \in \Re_+$, and $\vartheta \in \Re_+$, $f(k,z) = \mu + k^{\omega} z^{\vartheta}$ satisfies (T1)-(T3) and $g(I/k) = \alpha x/(\alpha + x)$, $x \equiv I/k$, $\alpha \in \Re_+$, satisfies (T4) and (T5).

Although the aggregate capital stock is a (positive) externality to each firm, its path is endogenously determined over time since the externalities $z(t)$ are realized as $k(t)$

[9] This usual concavity condition of the invest function becomes important for indeterminacy (see Section 4).

at each moment of time. More precisely, capital growth may occur in absence of population growth and exogenous technical changes.

Given the set assumptions on utility and technology, the level of consumption should be prevented from growing so fast that the agent's utility becomes infinite. Recall that $\dot{k} = kg(I/k)$ where $I = f(k,z) - c$. Assumption (T3) and (T5) imply that the maximum comprehensive consumption can grow at the rate $\alpha\rho$. Therefore, if $\alpha\rho < d$ then $U(C) < \infty$ for a given finite initial capital stock. Thus this generalized Brock-Gale's joint condition, together with the concavity assumptions on utility and technology, guarantees existence of a competitive equilibrium path with externalities.[10]

3. CONSUMPTION AND CAPITAL POLICY FUNCTION

In this section I characterize an equilibrium path in the competitive economy. Let us begin with denoting the initial capital stock $k(0)$ by $k_0 < \infty$, where $k_0 \in \Re_{++}$. A path of externalities is represented by Z, where for $0 \le t \le \infty$ the t-th period externality corresponding to Z is $z(t)$. Let E denote a set of externalities where any externality $z(t)$ is in $E \subseteq \Re_{++}$. In particular, for any T, ${}_T Z = \{Z : z(t) \in E, t \in [T,\infty)\}$. The competitive economy in the presence of externalities is formally (u, v, f, g, k_0), which satisfies the assumptions mentioned above. Then, the consumer maximizes his lifetime utility:

$$U(C) = \int_0^\infty u(c(t)) \exp[-\int_0^t v(c(s))ds]dt,$$

given the budget constraint, $\int_0^\infty q(t)c(t)dt \le q(0)k(0) + \int_0^\infty w(t)dt$, where $q(t)$ is the present valued price and $w(t)$ the present valued wage rate at time t. The firm maximizes its profit:

$$\int_0^\infty [q(t) f(k(t), z(t)) - w(t) - q(t)k(t)]dt.$$

At an equilibrium, $k(0) = k_0$, $\dot{k}/k(t) = g(I(t)/k(t))$ and $I(t) = f(k(t), z(t)) - c(t)$ for

[10] See von Neumann Equilibrium Theorem in Becker *et al.* (1989, p. 96). The formal proof is beyond the scope of this paper. Somewhat different from this model, a proof for existence of a growing equilibrium path with Romer-Lucas technology can be found in Suzuki (1996) where there is a growing optimal path with recursive preferences.

$t \in [0,\infty)$.

The general equilibrium growth problem, defined below as $PE(k_0, Z)$, in the economy (u, v, f, g, k_0) can be restated as the following: Let $y \equiv [f(k,z) - c]/k$. Then, for all $t \in [0,\infty)$, $PE(k_0, Z)$:

$$U(C) = \int_0^\infty u(c(t)) \exp[-\int_0^t v(c(s))ds]dt \quad \text{s.t.} \quad \dot{k} = k[g(y)], \quad k(0) = k_0, \quad z(t) \in E .$$

It is immediately apparent that $PE(k_0, Z)$ has a solution since this is a convex -maximization programming under the set of assumptions. Alternatively, I can define the value function $J(0, k_0 \mid Z)$ at time 0 such that

$$J(0, k_0 \mid Z) = \max\{U(C) : \dot{k} = k(t)g(y(t)); k(0) = k_0; z(t) \in E, t \in [0,\infty)\} .$$

I can therefore extend this definition to the value function $J(T, k_T|_T Z)$ at time T:

$$J(T, k_T|_T Z) = \max\ U({}_T C) = \int_T^\infty u(c(t)) \exp[-\int_T^t v(c(s))ds]dt ,$$

subject to $\dot{k} = kg(y)$, $k(T) = k_T$, $z(t) \in E$, $t \in [0,\infty)$, where $y \equiv [f(k,z) - c]/k$ and ${}_T C = \{C[T,\infty) : c(t) \in \Re_+, t \in [0,\infty)\}$.

Notice that the value function $J(t, k_t|_t Z)$ is time-independent in the sense that for any $s \in [0,\infty)$, $J(t, k_t|_t Z) = J(s, k_s|_s Z)$ provided that $k_t = k_s$, ${}_t Z = {}_s Z$. Thus $J(t, k_t|_t Z) \equiv J(k_t|_t Z)$.

Using the method of dynamic programming, I can derive the Bellman equation in the models with recursive preferences and with technology that exhibits increasing returns to scale in the economy. The derivation is in Appendix 1. That is,

PROPOSITION 1: Under Assumptions (F1)~(F4) and Assumptions (T1), (T2), (T4), and (T5), the Bellman equation for the competitive economy in the presence of externalities (u, v, f, g, k_0) is

$$0 = \max\{u(c(0)) - v(c(0))J(k_0 \mid Z) + kg(y)J_k(k_0 \mid Z)\} .$$

The proposition generalizes the Bellman equation in neoclassical models. In continuous time models with no externality and a constant discounting rate, say δ, the Bellman equation has been obtained as $0 = \max\{u(c) - \delta J(k) + \dot{k}J_k(k)\}$. By the nature of a recursive objective functional, the constant discounting rate δ is replaced by $v(c)$

in the Bellman equation of this model.

The necessary conditions for the Bellman equation for $PE(k_0, Z)$ are

$$u'(c) - v'(c)J(k_t|_t Z) - g'(y)J_k(k_t|_t Z) = 0 , \tag{N-1}$$

$$u''(c) - v''(c)J(k_t|_t Z) + [1/k]g''(y)J_k(k_t|_t Z) \le 0 . \tag{N-2}$$

(For the rest of this paper, time subscripts will be left out whenever this omission would cause no confusion.) The first condition (N-1) is obtained by differentiating the Bellman equation with respect to consumption, given the fixed capital stock. The second condition (N-2) is required for convexity of the Bellman equation in consumption. Condition (N-2) will then become one of the sufficient conditions for a solution to the Bellman equation for the competitive economy (u, v, f, g, k_0). In order to apply the implicit function theorem, the second condition will be strengthened to a strict inequality. The strengthened (N-2) will be used frequently in the following sections.

To establish the Hamiltonian equations, the shadow price of capital is defined as a marginal value of capital in terms of utility (Benveniste and Scheinkman (1982), Becker and Boyd (1992)). Here, the price of capital at time 0 is $p(0) \equiv J_k(k_0 \mid Z) = [1/g'(y)][u'(c) - v'(c)J(k_0 \mid Z)]$. The condition (N-1) is used in the second equality and an equilibrium path is assumed to be in the interior of the attainable set of the economy. The Principle of Optimality can be applied to extend this definition to the current value price of capital at time t as

$$p(t) \equiv J_k(k_t|_t Z) = [1/g'][u' - v'J(k_t|_t Z)],$$

where $k(t) = k_t$.

Let C be an equilibrium consumption path. Given the strengthened condition (N-2), the implicit function theorem allows this path to be written as a function of a fixed capital stock k_t:

$$0 = \max\{u(c(k_t)) - v(c(k_t))J(k_t|_t Z) + k_t g(y(t))J_k(k_t|_t Z)\},$$

where $y(t) = [f(k_t, z(t)) - c(k_t)]/k_t$. Supposing $J(k_t|_t Z)$ to be twice continuously differentiable,[11] the following equation is derived by differentiating this Bellman equation with respect to the capital stock k_t:

[11] The value function $J(K_t|_t Z)$ is continuously differentiable under the smoothness conditions on utility and technology, yet *twice* differentiability of the value function requires an additional set of assumptions. Araujo and Scheinkman (1977) reported conditions for the twice differentiablility of the value function in the time additive utility framework.

$$0 = kgJ_{kk} + [g + gD_1 f - g'y - v]J_k + [u' - v'J - g'J_k][\partial c / \partial k_t] ,$$

where J_{kk} , and $D_1 f$ denote $\partial J_k / \partial k$ and $\partial f(k,z)/\partial k$ respectively. Hence, with (N-1), (N-3), and $\dot{p}(t) = J_{kk}(k_t|_t Z)\dot{k}$, we have[12]

PROPOSITION 2: Under Assumptions (F1)~(F4) and Assumptions (T1), (T2), (T4), and (T5), and supposing the value function $J(k_t|_t Z)$ to be twice continuously differentiable, then the Hamiltonian equations with $y = [f(k,z) - c]/k$ in the economy (u, v, f, g, k_0) consist of

$$\dot{k} = kg(y) , \tag{H-1}$$
$$\dot{p} = -p[g(y) + g'(y)D_1 f(k,z) - g'(y)y - v(c)] . \tag{H-2}$$

Thus, the equations (H-1) and (H-2) determine the evolution of an equilibrium trajectory. I will show that, in Section 4, any trajectory determined by the equation (H-1) and (H-2) converges to a steady state, which automatically satisfies the transversality condition and constitutes an equilibrium. For completeness in listing necessary and sufficient conditions, the transversality condition is stated as

$$\lim D(t)p(t)k(t) = 0 \quad \text{as} \quad t \to \infty , \tag{H-3}$$

where $D(t) \equiv \exp[-\int_0^t v(c(s))ds]$.

At this point I examine what value of ζ in the condition (F3) guarantees positive prices for capital. In terms of the Arrow-Pratt measure for the curvature of the concave function u and v , $-v''/v \geq -u''/u' + \zeta$ implies that the discounting function is more concave by the measure $\zeta \geq 0$ than the felicity function. Then let the discounting function be called a 'ζ-concave function' with respect to the felicity function. The ζ-concave discounting function insures that prices $p(t)$ are positive over time.[13] In order to see this, suppose that $\zeta \geq |[g''J_k]/ku'|$. Assumption (F3) implies that $u' \geq v'[u''/v'' + g''J_k / kv']$. By the strengthened condition (N-2), a simple calculation

[12] Also see Park (2000) and Drugeon (1998). They derive independently the set of necessary conditions for an equilibrium in the same model as one in this paper.

[13] Since the assumption of ζ-concavity involves the endogenous value function, the author realizes that the assumption is less satisfactory.

yields $u' > v'J$. I thereby have[14]

LEMMA 1: Under Assumptions (F1)~(F4) and Assumptions (T1), (T2), (T4), and (T5), the price $p(t) \equiv J_k(k_t|_t Z)$ of capital in the economy (u, v, f, g, k_0) is strictly positive.

When the capital accumulation function produces one unit of additional capital by foregoing one unit of consumption, i.e., $g(y) = y$, the zero-concave discounting function is sufficient for a positive price of capital.

Deriving the consumption policy function, I digress briefly to derive the rate of time preference $\varphi(t)$ at time t. The derivation will reveal some key characteristics of the model, and it will be useful for simplifying the consumption policy function. First, I can measure the marginal utility with respect to an increment in consumption along a consumption path C about the time t by utilizing the concept of the Volterra derivative.[15] Then the marginal utility of consumption, $U_t(C)$ is equal to

$$[u'(c) - v'(c)U({}_tC)]\exp[-\int_0^t v(c(s))ds],$$

where $U({}_tC)$ is the current valued utility of ${}_tC$ at time t. Next, suppose ${}_tC$ is an equilibrium consumption path with the initial capital stock $k(t) = k_t$, so $U({}_tC) = J(k_t|_t Z)$. By using (N-1), $U_t(C)$ can also be written as $U_t(C) = D(t)g'(y(t))J_k(k_t|_t Z)$. Finally, the rate of time preference $\varphi(t)$ is defined as the negative of the logarithmic rate of change of marginal utility along the equilibrium path:

$$\varphi(t) \equiv -d[\log U_t(C)]/dt\big|_{\dot{c}=0} = -d[\log D(t)g'(y(t))J_k(k_t|_t Z)]/dt.$$

Finally, substitute $\dot{p}$ into $\varphi(t)$,

$$\varphi(t) = v(c) + \dot{p}/p + [gg''/g'][D_1 f + D_2 f - y].$$

Immediate observation shows that $\varphi(t)$ varies with $c(t)$ through the discounting function $v(c)$ and the capital accumulation function $g(y)$. This also varies with $k(t)$ through the price of capital $p(t)$, the output function $f(k, z)$, and the capital

[14] On the other hand, Epstein's (1987) approach requires the global monotonicity of utility by assuming that the marginal utility is strictly positive for each time.

[15] The volterra derivative is an operator, which firstly perturbs a consumption path by the magnitude less than $\varepsilon_1 > 0$ over a time interval of length $\varepsilon_2 > 0$ about t, and secondly lets ε_1 and ε_2 go to zero.

accumulation function $g(y)$. Furthermore, $\varphi(t)$ is dependent on the felicity function via a proportional price change $\dot{p}/p$. These relations of $\varphi(t)$ with $c(t)$ and $k(t)$ illustrate the nature of a recursive utility function in this model. It is also worth noting that $\varphi(t)$ is reduced to a constant $v(c)$ for all time t at a steady state.

Finally, we can derive the consumption policy function in terms of $\varphi(x)$ (see details in Appendix 2).

PROPOSITION 3: Given Assumptions (F1)~(F4) and Assumptions (T1), (T2), (T4) and (T5), and supposing the value function $J(k_t|_t Z)$ to be twice continuously differentiable, the consumption policy function for the economy (u, v, f, g, k_0) is defined by

$$\dot{c}\{[g''/kg'] + [[u'' - v''J]/[u' - v'J]]\} = \varphi(t) - v(c) + [kg/g']v' , \qquad \text{(H-4)}$$

where the rate of time preference $\varphi(t) = v(c) + \dot{p}/p + [gg''/g'][D_1 f + D_2 f - y]$.

Since the sign of $\{[g''/kg'] + [[u'' - v''J]/[u' - v'J]]\}$ cannot be assigned without more restrictions on the functions, changes in consumption are hard to predict with this model. On the other hand, if $g(y) = y$, then $\dot{c}\{[u'' - v''J]/[u' - v'J]\} = -D_1 f + v(c) + [f - c]v'$. Since $\varphi(t)$ also becomes equal to $v - v'\dot{k} = v - v'[f - c]$ with this specification of technology, the above equation can be further simplified as $\dot{c}\{[u'' - v''J]/[u' - v'J]\} = -D_1 f + \varphi$. This equation can be interpreted easily. Imposing the condition $u'' - v''J(k) < 0$, similar to the condition (N-2) in the case of this model, $[u'' - v''J(k)]/[u' - v'J]$ must be negative. Therefore, the higher output productivity induces less consumption and more investment now and then more consumption in the future. Conversely, impatience would imply more consumption now, and thus less in the future. Since v' is assumed to be positive, consumption will decrease when the changes in J are positive over time, and vice versa.[16] In addition to the condition $g(y) = y$, when the discounting function is constant, δ, independently on consumption, then $\dot{c}$ becomes $\dot{c}[-u''/u'] = D_1 f - \delta$. This is, the consumption policy function in Proposition 3 is a generalized version of a consumption policy function in the neo-classical growth model.

[16] Further interpretation of $[u'' - v''J]/[u' - v'J]$ can be found in Epstein (1987).

4. UNIQUENESS, LOCAL STABILITY, AND INDETERMINACY

We examine the local stability property for an equilibrium path in the competitive economy with externalities. Local stability can be demonstrated by adopting the Hamiltonian dynamics approach. This approach has been exploited in economies where preferences are additively time-separable and technology is concave (e.g., Levhari and Leviatan (1972)). The similar argument has also been adopted for economies with recursive preferences and a neo-classical technology (e.g., Epstein (1987)).

Now, we show the existence and uniqueness of a steady state in the economy (u, v, f, g, k_0).[17] First of all, a stationary path of the Hemiltonian equations for $PE(k_0, Z)$ is defined to be the triple (c, k, p) that satisfies $0 = \dot{k} = kg$ and $0 = -p[g + g' D_1 f - g' y - v]$. Then $f(k, z) = c$, $D_1 f(k, z) = v(c)$, and $z = k$ at the steady state.[18] The uniqueness of a steady state is shown as following: At a steady state,

$$-p[g(0) + g'(0) D_1 f - g'(0)[0] - v] = -p[D_1 f(k, k) - v] = 0 .$$

And then $D_1 f(k, k) = v(c) = v(f(k, k))$. The monotonicity conditions for the LHS and RHS of the equation ensure uniqueness of a steady state. That is, if $D_{11} f(k, k) + D_2 f(k, k) < 0$ and $v'[D_1 f(k, k) + D_2 f(k, k)] > 0$ then a steady state is unique. Assumptions (F2), (T2) guarantee that $v'[D_1 f(k, k) + D_2 f(k, k)] > 0$. The inequality: $D_{11} f(k, k) + D_2 f(k, k) < 0$ holds only when the effect of the (positive) externality is small enough. However, it is possible that multiplicity of steady states occurs when the effect of externalities dominates the degree of diminishing marginal product of the individual capital (also see Howitt and McAfee (1988)).

It is still necessary to prove that a steady state is a solution to $PE(k_0, Z)$. It can be accomplished by using a standard argument (see Epstein (Lemma 2, 1987)). A rough outline of argument is following: Let $\overline{C}$ be a stationary solution to the above problem. By the concavity of the felicity and discount function and concavity of the investment and output production function in k, for a given path of externalities, we can integrate equations (H-1), (H-2), and (H-3). Since all stationary paths must satisfy the transversality condition, $U(\overline{C}) \geq U(C)$ for any feasible stationary path C. Hence we have:

[17] The existence of a steady state may be proven by adopting the fixed point argument of Lucas and Stokey (1986, Section 7). An axiomatic approach for existence a balanced growth path is in Le Van and Vailakis (2005), and Farmer and Lahiri (2005).

[18] In fact, this model is ready for persistently growing balanced growth paths and the most of properties of steady states in the present paper can be re-obtained for a growing economy. For example, Palivos *et al.* (1997), and Zee (1997) extend to a long-run growth economy with a simple AK-technology.

PROPOSITION 4: Under Assumption (F1)~(F4) and Assumption (T1)~(T5) and supposing the value function $J(k_t|_t Z)$ to be twice continuously differentiable, if $|D_{11} f(k,k)| > |D_2 f(k,k)|$, then $PE(k_0, Z)$ has a unique stationary solution in the competitive economy (u, v, f, g, k_0).

I now implement the following steps for studying local stability: First, linearize Hamiltonian equations about a steady state (assuming that a locally unique steady state exists). Second, define the characteristic equations, which describe the properties of the original system of the equations. Third, if the system of equations satisfies the paired roots theorem, then the theorem can be used ascertain whether each system enjoys saddle-point properties.[19] In continuous time models the eigenvalues of the system of equations will have the saddle point property if the paired roots have the opposite signs (Theorem 1). If not, all roots have to be negative absolute stability (Theorem 2).

Now, the smoothness conditions on the functions in the competitive economy allows us to apply the implicit function theorem so that I can express the consumption function in terms of k and p. Given the Hamiltonian equations (H-1) and (H-2) for $PE(k_0, Z)$, the stationary equilibrium path implies that: $f(k,k) = c$ and $D_1 f(k,k) = v(c)$. Suppose that

$$H^k(k, p) \equiv kg(y),$$
$$H^p(k, p) \equiv -p[g(y) + g'(y)D_1 f(k, z) - g'(y)y - v(c)].$$

Then the Hamiltonian equations become $\dot{k} = H^k(k, p)$ and $\dot{p} = H^p(k, p)$. Let $H_k^k = \partial H^k / \partial k$, $H_p^k = \partial H^k / \partial p$, $H_k^p = \partial H^p / \partial k$, and $H_p^p = \partial H^p / \partial p$. The system of linearlized Hamiltonian equation (H-1) and (H-2) can be summarized as

$$\begin{bmatrix} \dot{k} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} H_k^k & H_p^k \\ H_k^p & H_p^p \end{bmatrix} \begin{bmatrix} k - \bar{k} \\ p - \bar{p} \end{bmatrix},$$

where $(\bar{k}, \bar{p})$ denote the steady state capital and its price. Let Ω be the matrix on the right hand side. The elements of the matrix are explicitly computed in Appendix 3. I can define the characteristic equation $\xi(\lambda)$ of the system of the linearized equations:

[19] The paired root theorem roughly states that if the roots are counted according to multiplicity, then eigenvalues of optimization problems occurs in pairs when the related matrix of the partial derivatives is not singular.

$\xi(\lambda) \equiv \det\left(\Omega - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$. Thus $\xi(\lambda) = \lambda^2 - (tr\Omega)\lambda + \det\Omega$. There are two different paired roots for $\xi(\lambda)$ such that λ_1, λ_2 are equal to $\frac{1}{2}\left[tr\Omega \pm \left[[tr\Omega]^2 - 4\det\Omega\right]^{1/2}\right]$ with $\lambda_1 > \lambda_2$. With the condition (N-2) and the positive prices for the capital, the signs of λ_1, λ_2 depend on the sign of $D_{11}f + D_2 f - v'[D_1 f + D_2 f]$ (see Appendix 3).

First, let us consider that case that $\det\Omega$ is negative. Appendix 3 shows that if $D_{11}f + D_2 f - v'[D_1 f + D_2 f] < 0$, $\det\Omega$ is negative and thus $\lambda_1 > 0 > \lambda_2$. The following theorem states that the eigenvalues of $\xi(\lambda)$ have the opposite sings and thereby the equilibrium path for $PE(k_0, Z)$ has the saddle point property at the stationary equilibrium $(\bar{k}, \bar{p})$.

THEOREM 1: Under Assumption (F1)~(F4) and Assumption (T1)~(T5), and supposing the value function $J(k_t|_t Z)$ to be twice continuously differentiable, if the discounting function is ζ -concave with respect to the utility function, the determinant of the matrix of the system of linearized Hamiltonian equations Ω is not singular and

$$\left|\left[D_{11}f/[D_1 f + D_2 f]\right] - v'\right| \geq \left|D_2 f/[D_1 f + D_2 f]\right|,$$

then the stationary equilibrium path for the competitive economy (u, v, f, g, k_0) with externalities is locally (saddle) stable.

Now, remember that $D_{11}f(k,z)$ is negative by (T2) and $v'(c)$ is positive by (F2). Hence, when the economy has either no or negative spillover effects of private production, but small than the productivity of private production, i.e., $D_2 f \leq 0$ and $D_1 f + D_2 f > 0$, $\det\Omega < 0$. Therefore $\lambda_2 < 0 < \lambda_1$ and then the saddle property holds at the steady state. In a linear model where $D_{11}f(k,z) = 0$, an economy can be locally stable as long as a consumer becomes impatient as the economy develops. More interestingly, if there are the positive effects from externality, i.e., $D_2 f > 0$, these effects have to be small-with respect the change in both the individual marginal productivity of capital and the rate of time preferences-in order to ensure that $\det\Omega < 0$. It is now clear that an increase in impatience and the concavity of the productivity of the private production function are stabilizers, while a (positive) externality is a destabilizer.

The underlining logic is rather simple. In endogenous growth models, externalities play a main role to generate economy-wide increasing returns by overriding diminishing returning of private investment. This substitution effect always destabilizes an economy when a consumer has a time-separable preference (see Romer (1986), Lucas (1988)). In contrast, the present economy can be stable if a consumer becomes impatient as he/she becomes rich. More precisely, if the intertemporal substitution effect from in consumption choice between periods dominates the substitution effect from investment plus a usual income effect, then the economy is locally stable.

We further observe that the conditions for uniqueness of the steady state (Proposition 4) ensure local convergence of an equilibrium. Therefore there is a unique equilibrium path converges to the unique steady state. This is consistent with the result of Benhabib and Gali (1995), i.e., as long as the uniqueness of the steady state is preserved; the uniqueness of the (transitional) equilibrium is also preserved even in the present of market imperfections. We conclude

COROLLARY 1: Under Assumption (F1)~(F4) and Assumption (T1)~(T5) and supposing the value function $J(k_t|_t Z)$ to be twice continuously differentiable, if $|D_{11}f(k,k)| > |D_2 f(k,k)|$, then the competitive equilibrium path in the economy (u, v, f, g, k_0) is unique and converges to the unique steady state.[20]

But, when $\det\Omega > 0$ the saddle point property fails. There are two possibilities in this case: Either $tr\Omega > 0$ or $tr\Omega < 0$. First, $\lambda_1 > \lambda_2 > 0$ whenever $tr\Omega > 0$ and $\det\Omega > 0$, and thus an equilibrium diverges from the steady state. (I may leave further analysis for this case to the readers.) Secondly, when $tr\Omega < 0$ in the case of $\det\Omega > 0$, the economy is absolutely stable because both λ_2, λ_1 have a negative real part (if roots are complex).[21] Therefore, we recover the local stability of the steady state as the following:

THEOREM 2: Under the same hypotheses as Theorem 1, except the inequality is replaced by

$$|D_1 f + D_2 f| < |[Apg'' D_2 f]/k|, \text{ and } |D_{11} f/[D_1 f + D_2 f] - v'| < |D_2 f/[D_1 f + D_2 f]|,$$

where $A \equiv [[pg''/k] + u'' - v''J]^{-1}$, the stationary equilibrium path for the competitive economy (u, v, f, g, k_0) with externalities is absolutely stable.

The first inequality condition in Theorem 2 is obtained from the sufficient condition for $tr\Omega < 0$, in the presence of the positive externality and adjustment cost of investment. Clearly, when there is no externality the absolute stability property does not

[20] Park (2000) also shows the global turnpike property under the smoothness condition on the consumption and capital policy function as in the present model with recursive preferences.

[21] The equilibrium path can be cyclical in transaction with additional condition that $[tr\Omega]^2 < 4\det\Omega$. Since the trajectory absolutely converges to a steady state, this path also satisfies the transversality condition.

hold. We also notice that, in addition to the change of impatience and the concavity of the private production function, the strong concavity of investment function works as a stabilizer.

Also, the absolute stability suggests possibility of the indeterminacy of equilibrium paths in a growing economy (the symposium issue in Journal of Economic Theory (1994)). Theorem 2 provides the sufficient condition for indeterminacy of steady states in the sense that there is a continuum of equilibrium paths associated to the same economic fundamentals including the same initial stock of capital. That is,

COROLLARY 2: Under the same hypotheses as Theorem 2, there exists indeterminacy of steady state equilibrium paths in the competitive economy (u, v, f, g, k_0).

First, the presence of externality is necessary for the result. When there is no externality, i.e., $D_2 f = 0$, the both inequality conditions are violated and we thereby go back to the saddle stable case in theorem 1. Moreover, a continuum of equilibria would occur only under the condition that $|D_{11} f| < |D_2 f|$, which is consistent with the conditions of Corollary 2. Second, the adjustment cost for investment should not be zero for indeterminacy.[22] That is, this Corollary is consistent with the results of Benhabib and Peril (1994) that externalities (even though large) are not sufficient to generate indeterminacy. Third, since there is a continuum of transitional paths near a steady-state equilibrium path a sunspots equilibrium can be constructed and thereby a self-fulfilling equilibrium can exhibit business cycles. Finally, although the presence of externality is essential for the indeterminacy, the indeterminacy does not require a sufficiently large magnitude of the effect of externalities as long as the adjustment cost, i.e., g'', is large enough.[23]

5. CONCLUDING REMARKS

This paper establishes the sufficient condition for local stability of a competitive economy with recursive preferences and increasing returns technology. It is shown that that the positive externalities from social capital to private production cause instability and multiplicity in the competitive economy. Since in a model with the Romer-Lucas technology, the competitive equilibrium is not socially optimal, it is interesting to ask

[22] Our indeterminacy result differs from one in Drugeon (1998), who introduces consumption externality on the discounting function.

[23] Notably, Benhabib and Farmer (1994) calibrated a model, using the size of externalities in the US economy, whose steady state is unique and locally stable.

whether, given a set of sufficient conditions for existence of a socially optimal path, is that path locally stable. It seems that unstable or cyclical behavior of the optimal path can be induced by the very nature of an increasing returns technology.

However, the stability of the optimal path may be restored by the same reason in the present paper that the discount rate increases rapidly with the presence of adjustment cost in investment. But this absolute stability results also causes indeterminacy of equilibrium paths and a self-fulfilling equilibrium can generate business cycles around the steady state equilibrium.

There are still interesting questions remaining in this model. For example, the relation between business cycles, and sunspots is worthwhile to explore in a model with endogenous preferences and technological changes. Furthermore, necessary and sufficient conditions for a continuum of equilibrium paths or of conditions for multiple stationary equilibria remain to be found. Furthermore, it will be interesting to see growth cycles in growth models of recursive preferences.

Appendix 1.

Using the methods of dynamic programming, we can easily obtain the Bellman equation: According to the Principle of Optimality and given the historically determined initial capital stock k_T at time T, a portion of the equilibrium path ${}_T C$, where $T > 0$, must be an equilibrium path. Therefore the future consumption path ${}_T C$ is independent of the manner in which the system arrived at the capital stock k_T. But future decisions are affected via externalities Z and the discount factor $D(T) \equiv \exp[-\int_0^T v(c(s))ds]$, due to a recursive objective functional in the model. Thus $J(k_0 \mid Z)$ can be broken up as

$$\max\{\int_0^T u(c(t))\exp[-\int_0^t v(c(s))ds]dt + \max\int_T^\infty u(c(t))\exp[-\int_0^t v(c(s))ds]dt\}$$
$$= \max\{\int_0^T u(c(t))\exp[-\int_0^t v(c(s))ds]dt + \exp[-\int_0^T v(c(s))ds]J(k_T|_T Z)\}.$$

Since $D(0) = 1$, the above equation becomes

$$0 = \max\{\int_0^T u(c(t))D(t)dt + [D(T) - D(0)]J(k_T|_T Z) + J(k_T|_T Z) - J(k_0 \mid Z)\}.$$

By dividing the whole equation by T and letting $T \to \infty$, we can conclude

$$0 = \max\{u(c(0) - v(c(0)J(k_0 \mid Z) + kg(y)J_k(k_0 \mid Z)\}.$$

Appendix 2.

In order to derive the consumption policy function, differentiate $p(t)$ with respect to time t:

$$\dot{p}(t) = \{[g''/k[g']^2[u' - v'J] + [1/g'][u'' - v''J]\}\dot{c}$$
$$+ \{[-g''/k[g']^2][u' - v'J][D_1 f + D_2 f - y] - [1/[g']^2][u' - v'J]v'\}\dot{k}.$$

By substituting the condition (H-2) for $\dot{p}(t)$ and performing some algebraic manipulations, the result is

$$\dot{c}\{[g''/kg'] + [[u'' - v''J]/[u' - v'J]]\}$$
$$= -[g + g'D_1 f - g'y - v] + [gg''/g'][D_1 f + D_2 f - y] + [kg/g']v'.$$

By simplifying the consumption policy function $\dot{c}$ through substituting $\varphi(t)$, I can conclude

$$\dot{c}\{[g''/kg'] + [[u'' - v''J]/[u' - v'J]]\} = \varphi(t) - v(c) + [kg/g']v'.$$

Appendix 3.

Applying a standard method can derive the system of linearized Hamiltonian equations. Recall that

$$H^1 = kg(y),$$
$$H^2 = -p[g(y) + g'(y)D_1 f(k,z) - g'(y)y - v(c)],$$

where $y = [f(k,z) - c]/k$. Differentiate $H^1(k,p)$ and $H^2(k,p)$ with respect to k. Then,

$$H_k^1 \equiv \partial H^1/\partial k = g + g'[D_1 f + D_2 f - y] - g'[\partial c/\partial k],$$
$$H_k^2 \equiv \partial H^2/\partial k = -p[g''/k][D_1 f - y][D_1 f + D_2 f - y]$$
$$- pg'[D_{11} f + D_2 f] + p[[g''/k][D_1 f - y] + v'][\partial c/\partial k].$$

Differentiate $H^1(k,p)$ and $H^2(k,p)$ with respect to p. Then,

$$H_p^1 \equiv \partial H^1/\partial p = -g'[\partial c/\partial p],$$
$$H_p^2 \equiv \partial H^2/\partial p = -[g + g'D_1 f - g'y - v] + p\big[[g''/k][D_1 f - y] + v'\big][\partial c/\partial p].$$

In order to find $\partial c/\partial k$ and $\partial c/\partial p$ I differentiate the condition (N-1). Then, since the related matrix of partial derivatives is not singular,

$$\partial c/\partial k = p\big[[[D_1 f + D_2 f]g''/k] + v'\big]\big[[pg''/k] + u'' - v''J\big]^{-1},$$
$$\partial c/\partial p = \big[[pg''/k] + u'' - v''J\big]^{-1}.$$

Substitute $\partial c/\partial k$ and $\partial c/\partial p$ into H_k^1, H_p^1, H_k^2 and H_p^2, and then evaluate at the steady state. Let

$$A \equiv \big[[pg''/k] + u'' - v''J\big]^{-1},$$
$$B \equiv p\big[[[D_1 f]g''/k] + v'\big],$$
$$C \equiv p\big[[[D_1 f + D_2 f]g''/k] + v'\big].$$

Then, the system of linearized Hamiltonian equations (H-1), (H-2) can be summarized as

$$\begin{bmatrix} \dot{k} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} D_1 f + D_2 f - AC & -A \\ -p[D_{11} f + D_2 f] - [D_1 f + D_2 f][B - pv'] + ABC & AB \end{bmatrix} \begin{bmatrix} k - \bar{k} \\ p - \bar{p} \end{bmatrix},$$

where $(\bar{k}, \bar{p})$ denote the steady state consumption and the price of capital. Hence, Ω is the matrix of the right hand side of the above expression. Therefore, $tr\Omega = D_1 f + D_2 f - A[pg''D_2 f/k]$, and $\det\Omega = Ap\big[v'[D_1 f + D_2 f] - D_{11} f - D_1 f\big]$.

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